6.1 Stability and the Phase Plane

we will look at <u>autonomous systems</u>

$$\left.\begin{aligned}\frac{dx}{dt} &= F(x,y)\\ \frac{dy}{dt} &= G(x,y)\end{aligned}\right\}$$

F, G don't depend on t explicitly (even though x, y are implicit functions of t)

they can be linear and homogeneous, for example

$$\left.\begin{aligned}\frac{dx}{dt} &= -2x - y\\ \frac{dy}{dt} &= -x - 2y\end{aligned}\right\}$$

usual $\vec{x}' = A\vec{x}$

or nonhomogeneous and linear. for example

$$\frac{dx}{dt} = -2x - y + \boxed{5}$$

$$\frac{dy}{dt} = -x - 2y + \boxed{4}$$

nonhomogeneous

or nonlinear (homogeneous or non), for example

$$\frac{dx}{dt} = y^2$$

$$\frac{dy}{dt} = -x$$

a <u>critical point</u> (x, y) is where $\frac{dx}{dt} = \frac{dy}{dt} = 0$

→ it is an <u>equilibrium</u> solution

all linear homogeneous systems $\vec{x}\,' = A\vec{x}$ have only <u>one</u> critical point → $(0, 0)$

for $\vec{x}\,' = A\vec{x} + \vec{g}$ where $\vec{g}$ is a <u>constant</u> vector, there is only one critical point but moved to a different location, but the phase diagram remains the same as $\vec{x}\,' = A\vec{x}$

let's look at $\begin{aligned} x' &= -2x - y \\ y' &= -x - 2y \end{aligned} \rightarrow \vec{x}' = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \vec{x}$

improper nodal sink w/ critical pt $(0,0)$

now $\begin{aligned} x' &= -2x - y + 5 \\ y' &= -x - 2y + 4 \end{aligned} \rightarrow \vec{x}' = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

crit. pt: $x' = 0 \qquad -2x - y + 5 = 0$

$y' = 0 \qquad -x - 2y + 4 = 0$

$\vdots$

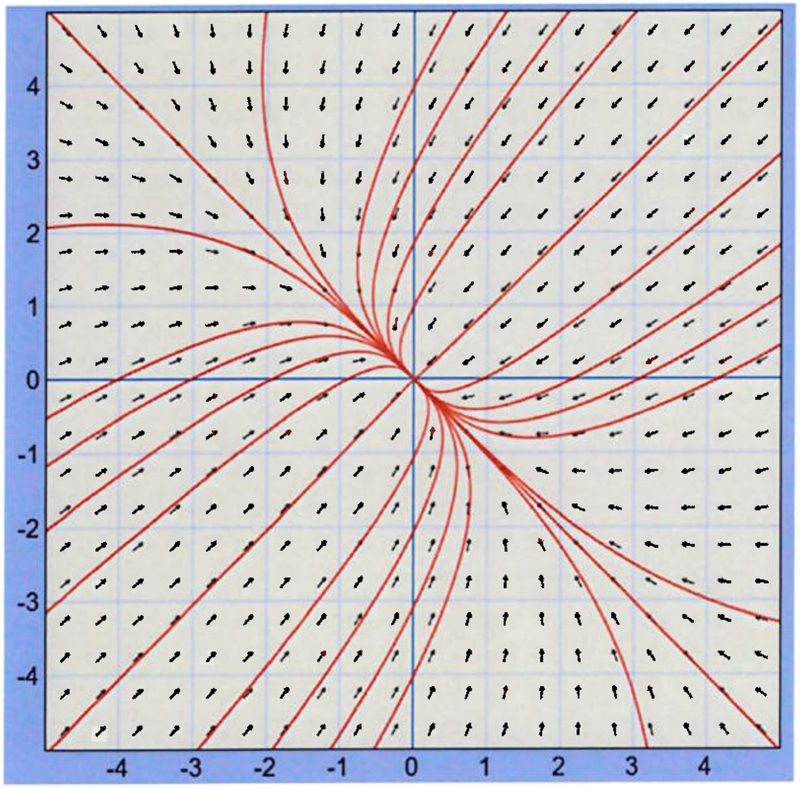
$x = 2, \quad y = 1$

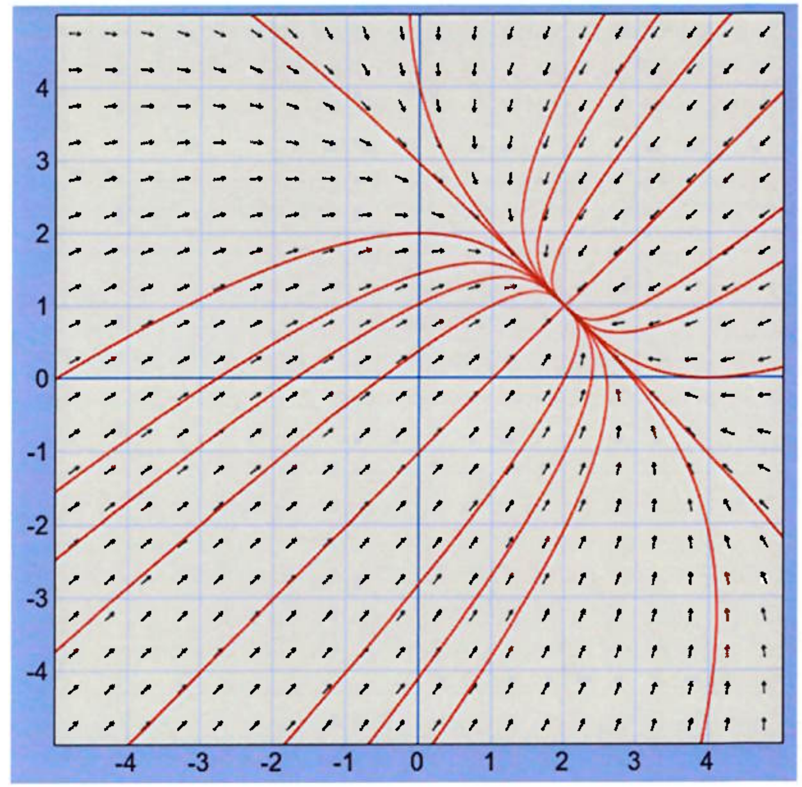
$(2, 1)$

phase diagram is the same as $\vec{x}' = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \vec{x}$

but shifted to be centered at $(2,1)$

instead of $(0,0)$





$$x' = -2x - y$$
$$y' = -x - 2y$$

Critical point $(0,0)$

$$x' = -2x - y + 5$$
$$y' = -x - 2y + 4$$

Critical point $(2,1)$

Notice the phase portraits are identical but the system with constant nonhomogeneous term has the “origin” shifted a nonlinear system can have multiple critical pts.

for example, $x' = x(2-y)$

$y' = y(x-3)$

crit. pts: $(0,0), (3,2)$

another example, $x' = x^2 - y - 1$

$y' = x - y - 1$

$x' = 0 \rightarrow y = x^2 - 1$

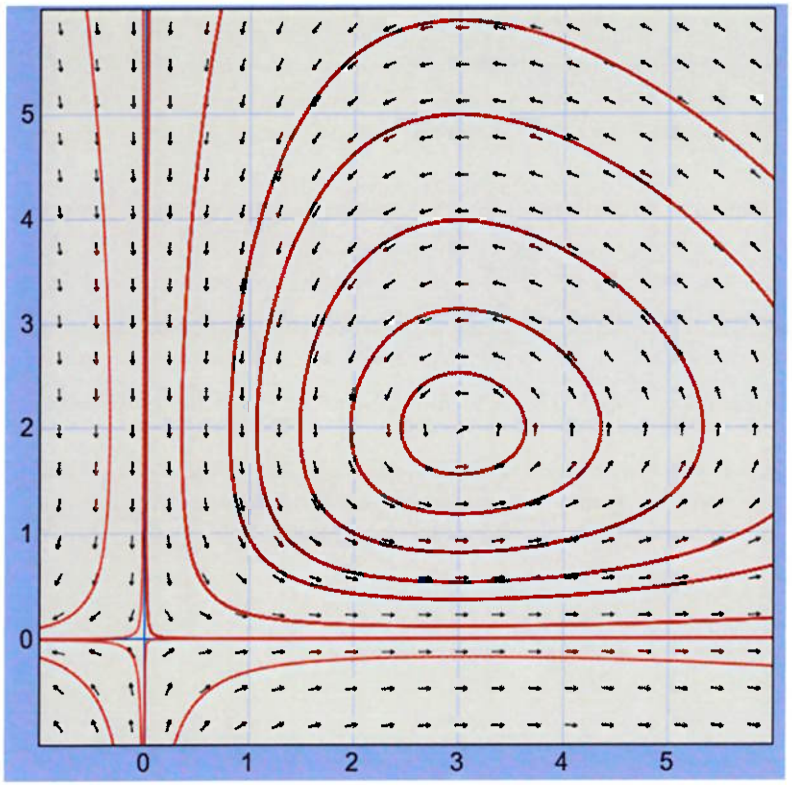
$y' = 0 \rightarrow x - (x^2-1) - 1 = 0$

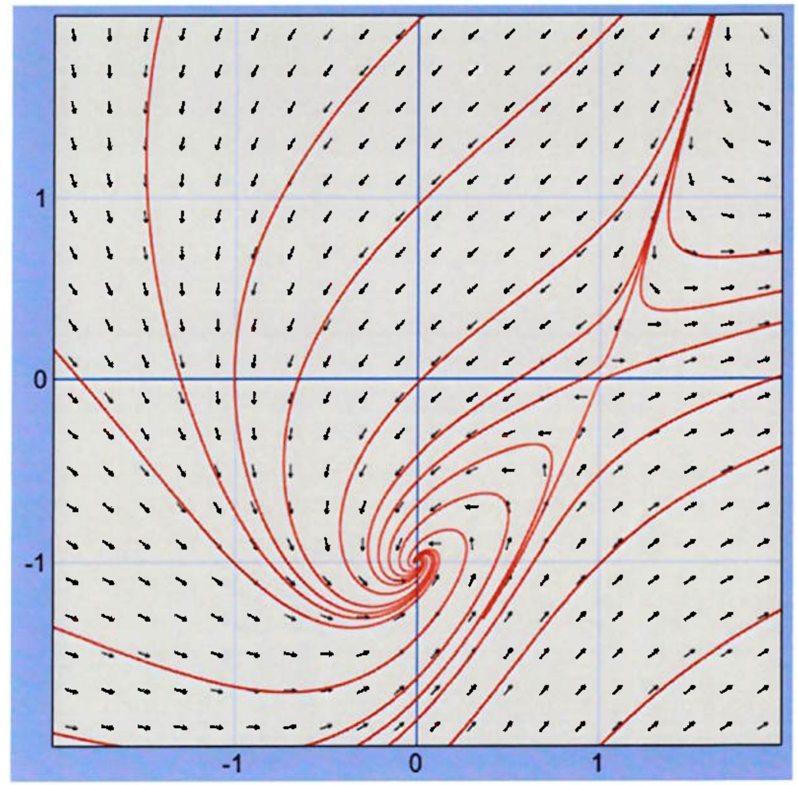
$x = 0, \quad x = 1$

$y = -1, \quad y = 0$

crit. pts $(0,-1), (1,0)$

phase diagram of a nonlinear is generally complicated but near each crit. pt the phase diagram will resemble that of a linear sys.





$$x' = x(2 - y)$$
$$y' = y(x - 3)$$

Critical points $(0,0)$, $(3,2)$

$$x' = x^2 - y - 1$$
$$y' = x - y - 1$$

Critical points $(0, -1)$, $(1,0)$

Notice each critical point resembles a center, improper/proper nodal source/sink saddle point, or spiral source/sink

Stability of critical pts

if solutions near a crit. pt stay near a crit. pt, the crit. pt is said to be stable (center)

if solutions fall into a crit. pt. → asymptotically stable

(any sort of sink)

if solutions run away → unstable

(any sort of source ~~a~~ or saddle pt)

general idea for nonlinear sys: look at what happens near each critical pt

nonlinear sys are ~~usually~~ often hard to solve:

$$x' = \tan(xy) + e^{y^2}$$
$$y' = \sin^2(x+y)$$

there are cases where we can turn the system into a differential eq. we can solve

$$\left.\begin{aligned}\frac{dx}{dt} &= -y^2 \\ \frac{dy}{dt} &= x\end{aligned}\right\} \quad \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{x}{-y^2}$$

separable

$$-y^2\,dy = x\,dx$$

$$-\frac{1}{3}y^3 = \frac{1}{2}x^2 + C$$

$$\frac{1}{2}x^2 + \frac{1}{3}y^3 + C = 0$$

implicit solution (can be used to graph each solution curve on the phase plane)